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Necessary and sufficient conditions for (i) Weyl, (ii) Riemann–Cartan connections

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Spaces with semi-metric connections (which include metric, Weyl and Riemann–Cartan connections), defined by $\nabla_c h_{ab}=h_{ab}\lambda_c$, necessarily satisfy an algebraic relationship of the type $h_{ai}\check{R}^i{}_{bcd}+h_{bi}\check{R}^i{}_{acd}=0$, where h_{ab} is a metric tensor, and $\check{R}^a{}_{bcd}$ is related to the curvature tensor $R^a{}_{bcd}$ of the connection by $\check{R}^a{}_{bcd}=R^a{}_{bcd}-\frac{1}{4}\delta^a_b R^i{}_{icd}$. It is shown—in a four-dimensional space-time, for almost all curvature tensors—that this algebraic relationship is also a sufficient condition for the local existence of a curvature tensor of a semi-metric connection. Generalisations of this result, involving a tensor more general than the curvature tensor, are also given.

Keywords: *curvature tensor, Weyl connections, Riemann–Cartan connections*
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1. Introduction

In trying to determine sufficient conditions for a symmetric connection Γ to be metric, i.e.,

$$\nabla_c g_{ab}=0\,, \tag{1.1}$$

where the metric tensor g_{ab} is assumed symmetric and where ∇ is related to Γ in the usual way, standard procedure would be to consider the whole set of integrability conditions, beginning with the first,

$$g_{ai}R^i{}_{bcd}+g_{bi}R^i{}_{acd}=0\,, \tag{1.2}$$

where $R^a{}_{bcd}$ is the curvature tensor of the symmetric connection Γ.

In ref. [1] we have shown—in a four-dimensional space–time for *almost all* curvature tensors of symmetric connections—that this first integrability condition (1.2) *alone* is a sufficient condition for the local existence of a Riemann tensor (of a symmetric metric connection).

In ref. [2] we obtained a more general version of this result. We showed that a sufficient condition for a symmetric connection which is volume preserving [i.e.,

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$R^a{}_{acd}(\Gamma)=0$] to be metric, depended on the existence of a Bianchi tensor $B^a{}_{bcd}$ with respect to this symmetric connection Γ which satisfies

$$h_{ai}B^i{}_{bcd}+h_{bi}B^i{}_{acd}=0 . \tag{1.3}$$

[A "Bianchi tensor $B^a{}_{bcd}$ with respect to a symmetric connection Γ" was defined to be a four-tensor whose indices have the same symmetry properties as the curvature tensor of a symmetric connection, i.e.,

$$B^a{}_{bcd}+B^a{}_{bdc}=0 , \tag{1.4}$$

$$B^a{}_{[bcd]}=0 , \tag{1.5}$$

and which also satisfies the same Bianchi-type equation as the curvature tensor of a symmetric connection,

$$\nabla_{[e}B^a{}_{|b|cd]}=0 .] \tag{1.6}$$

This result was also shown to be valid for *almost all* Bianchi tensors of symmetric connections, in a four-dimensional space–time.

If we do not impose this volume-preserving condition on Γ then (1.3) alone is insufficient to guarantee that the connection is metric; rather it was found in ref. [2] that the symmetric connection Γ and metric h_{ab} are related according to

$$\nabla_c h_{ab}=h_{ab}\lambda_c . \tag{1.7}$$

This is the symmetric connection introduced into general relativity by Weyl [3], and discussed in the classical texts [4,5] and more recently in ref. [6]. The results in [2] would lead us to suspect that there may be some simple basic underlying result for Weyl connections, so that the results which we have already found for metric connections are just special cases of some more general result.

In the following section we show that this is indeed so; we show that the existence of a generalised Bianchi tensor $B^a{}_{bcd}$ with respect to a symmetric connection Γ which satisfies the algebraic constraint equation (1.3) is a necessary and sufficient condition that the symmetric connection Γ be a Weyl connection. [A "generalised Bianchi tensor $B^a{}_{bcd}$ with respect to a symmetric connection Γ" is defined to be a four-tensor whose indices have the symmetry properties (1.4) and which also satisfies the Bianchi-type eq. (1.6) with respect to the symmetric connection Γ.] As a special case of this result we obtain necessary and sufficient conditions on a curvature tensor of a symmetric connection in order that it come from a Weyl connection.

In section 3 we show that this type of property is not restricted to connections which are symmetric; analogous results exist for semi-metric connections [4] (satisfying equations of the form (1.7) with connections which are asymmetric, i.e., include torsion). As a special case of these, we consider Riemann–Cartan connections [7] in section 4.

Each of the results in this paper is subject to the condition that a certain matrix has maximal rank; in four dimensions the matrix is square and its size is 36×36. In section 5 we show that this condition is almost always satisfied in four-dimensional space–time; it is only in some very special circumstances that the rank condition is violated. In section 6 we discuss some related problems and directions in which these results can be developed.

2. Conditions for Weyl connections

We shall assume that **M** is a smooth four-dimensional manifold on which there is a symmetric connection Γ, associated covariant derivative ∇, and curvature tensor components $R^a{}_{bcd}$ defined in the usual way. In addition on **M** there is defined a non-degenerate symmetric Lorentz metric with components h_{ab} which is not related *a priori* to the symmetric connection Γ.

A Weyl connection [3] is one in which the vector λ_a links the symmetric connection Γ and metric h_{ab} according to

$$\nabla_c h_{ab} = h_{ab}\lambda_c \,. \tag{2.1}$$

The first integrability condition of (2.1) is easily seen to be [#1]

$$h_{ai}R^i{}_{bcd} + h_{bi}R^i{}_{acd} = 2h_{ab}\nabla_{[d}\lambda_{c]} \,. \tag{2.2}$$

The usual way to find a sufficient condition for (2.1) is to differentiate (2.2) to get the second integrability condition, and to keep repeating the procedure until the complete set of integrability conditions are found [5]. However, we wish to prove that, usually, the first set of integrability conditions (2.2) is *by itself* sufficient to imply (2.1).

The analogous result for metric connections in ref. [1] was later found in ref. [2] to be a special case of a more general result, dependent on the existence of a "Bianchi tensor". So we will first of all obtain the more general result—in terms of a Bianchi tensor—for Weyl connections, and then specialise it to conditions on the curvature tensor.

So—by analogy with the result in ref. [2]—in order to find sufficient conditions that Γ be a Weyl connection, we assume that there exists on **M** a "Bianchi tensor $B^a{}_{bcd}$ with respect to the symmetric connection Γ". This is a four-tensor $B^a{}_{bcd}$ which satisfies the algebraic identities

$$B^a{}_{bcd} + B^a{}_{bdc} = 0 \,, \tag{2.3a}$$

$$B^a{}_{[bcd]} = 0 \,, \tag{2.3b}$$

and also satisfies the Bianchi-type equation

[#1] We define the curvature tensor of a symmetric connection by $R^a{}_{bcd}V^b = (\nabla_c\nabla_d - \nabla_d\nabla_c)V^a$.

$$\nabla_{[e}B^a{}_{|b|cd]}=0 \tag{2.4}$$

with respect to the symmetric connection Γ. We further assume, in this case, that the Bianchi tensor $B^a{}_{bcd}$ satisfies the additional algebraic relation

$$h_{ai}B^i{}_{bcd}+h_{bi}B^i{}_{acd}=h_{ab}b_{cd}\,, \tag{2.5}$$

where b_{cd} $(=-b_{dc})$ is a bivector defined on $\mathbf{M}$, which is easily seen to satisfy

$$B^a{}_{acd}=2b_{cd}\,. \tag{2.6}$$

It follows immediately, from the fact that $B^a{}_{bcd}$ is a Bianchi tensor satisfying the Bianchi equations, that b_{cd} can be written in terms of a vector b_a,

$$b_{cd}=\nabla_{[d}b_{c]}\,. \tag{2.7}$$

b_a is of course defined up to an additive gradient.

If we now make the substitution

$$B^a{}_{bcd}=\breve{B}^a{}_{bcd}+\tfrac{1}{4}\delta^a_b B^i{}_{icd} \tag{2.8}$$

into (2.5), and use (2.6), we obtain

$$h_{ai}\breve{B}^i{}_{bcd}+h_{bi}\breve{B}^i{}_{acd}=0\,. \tag{2.9}$$

We note that $\breve{B}^a{}_{bcd}$ shares the index symmetry (2.3a) with $B^a{}_{bcd}$, satisfying

$$\breve{B}^a{}_{bcd}+\breve{B}^a{}_{bdc}=0\,, \tag{2.10a}$$

and a more complicated version of (2.3b),

$$\breve{B}^a{}_{[bcd]}=-\tfrac{1}{2}\delta^a_{[b}\nabla_d b_{c]}\,. \tag{2.10b}$$

Also $\breve{B}^a{}_{bcd}$ satisfies the Bianchi-type equation,

$$\nabla_{[e}\breve{B}^a{}_{|b|cd]}=0 \tag{2.11}$$

with respect to the symmetric connection Γ.

Let us now define a "generalised Bianchi tensor with respect to the symmetric connection Γ" to be a four-tensor $B^a{}_{bcd}$ which satisfies the algebraic conditions (2.3a) and also satisfies the Bianchi-type eq. (2.4) in the symmetric connection Γ. [A "Bianchi tensor with respect to the symmetric connection Γ"—defined earlier—picks out that subclass of generalised Bianchi tensors which satisfies the additional algebraic condition (2.3b).] The tensor $\breve{B}^a{}_{bcd}$ defined by (2.8) is therefore—by virtue of (2.10a) and (2.11)—a generalised Bianchi tensor with respect to the symmetric connection Γ.

We now follow the same procedure as in refs. [1] and [2]. Differentiating (2.9) and using the Bianchi equations (2.11) we find

$$\breve{B}^i{}_{b[cd}\hat{Q}_{e]ai}+\breve{B}^i{}_{a[cd}\hat{Q}_{e]bi}=0\,, \tag{2.12}$$

where [#2]

$$\hat{Q}_{cab} = -\nabla_c h_{ab} . \tag{2.13}$$

Although the metric tensor h_{ab} is not associated with the connection Γ in the usual way, we can still use it to lower (and use h^{ab} to raise) indices—but care must be taken to avoid ambiguities and errors since h_{ab} is not constant with respect to ∇.

So we find ourselves with exactly the same algebraic problem as in refs. [1] and [2]. At first sight the system of equations (2.12) appears—in four dimensions—to be forty real equations in the forty real unknowns $\hat{Q}_{cab}(=\hat{Q}_{cba})$. However, four linear combinations of these equations are easily seen to be identically zero, since

$$h^{ab}\breve{B}^i{}_{a[cd}\hat{Q}_{e]bi} \equiv 0 . \tag{2.14}$$

Also $\hat{Q}_{cab}$ cannot be uniquely determined by (2.12) since the system is unchanged under the transformation

$$\hat{Q}_{cab} \to \hat{Q}_{cab} + l_c h_{ab} \tag{2.15}$$

for an arbitrary vector l_c. So in fact the system (2.12) reduces to a square homogeneous system of thirty-six equations

$$\breve{B}^i{}_{b[cd}\tilde{Q}_{e]ai} + \breve{B}^i{}_{a[cd}\tilde{Q}_{e]bi} = 0 \tag{2.16}$$

in the thirty-six unknowns $\tilde{Q}_{cab}$ [#2],

$$\tilde{Q}_{cab} = \hat{Q}_{cab} - \tfrac{1}{4} h_{ab}\hat{Q}_{ci}{}^{i} . \tag{2.17}$$

This means that the mapping

$$\tilde{Q}_{cab} \to \breve{B}^i{}_{b[cd}\tilde{Q}_{e]ai} + \breve{B}^i{}_{a[cd}\tilde{Q}_{e]bi} \tag{2.18}$$

can be represented by a 36×36 matrix ${}^{s}A_{\breve{B}}$ consisting of components of the generalised Bianchi tensor $\breve{B}^a{}_{bcd}$. If this matrix has maximal rank then the only solution to eq. (2.16) is the trivial one,

$$\tilde{Q}_{cab} = 0 , \tag{2.19}$$

which is equivalent to

$$\nabla_c h_{ab} = -\tfrac{1}{4} h_{ab}\hat{Q}_{ci}{}^{i} . \tag{2.20}$$

The integrability condition of (2.20) gives

$$\tfrac{1}{2} h_{ab}\nabla_{[c}\hat{Q}_{d]i}{}^{i} = h_{ai}R^i{}_{bcd} + h_{bi}R^i{}_{acd} , \tag{2.21}$$

which implies

[#2] We follow the same convention in our definitions $\hat{Q}_{cab}$ and $\tilde{Q}_{cab}$ as in ref. [2].

$$2\nabla_{[c}\hat{Q}_{d]i}{}^{i}=R^{a}{}_{acd}\,, \tag{2.22}$$

and so in this case—unlike in refs. [1] and [2]—it does not follow automatically that $\hat{Q}_{dij}h^{ij}$ is a gradient vector, nor that there exists a metric which is annihilated by ∇. So the connection Γ is, in this case, of a more general type than a metric connection, and is given from (2.20) by

$$\nabla_c h_{ab}=h_{ab}\lambda_c \tag{2.23}$$

or

$$\Gamma^a_{bc}=\left\{{a \atop bc}\right\}-\tfrac{1}{2}\delta^a_b\lambda_c-\tfrac{1}{2}\delta^a_c\lambda_b+\tfrac{1}{2}h_{bc}\lambda_i h^{ia}\,, \tag{2.24}$$

where

$$\lambda_c=-\tfrac{1}{4}\hat{Q}_{ci}{}^{i} \tag{2.25}$$

and $\left\{{a \atop bc}\right\}$ are the Christoffel symbols of the metric h_{ab}. So the symmetric connection Γ is a Weyl connection [3], given in terms of a metric h_{ab}, and a vector λ_a.

We can therefore state the following result:

Theorem 1. *At any point in a co-ordinate domain U of* **M**, *a necessary and sufficient condition that a symmetric connection Γ be a Weyl connection, i.e., that Γ satisfies (for some metric h_{ab} and some vector λ_a)*

$$\nabla_c h_{ab}=h_{ab}\lambda_c\,, \tag{2.26}$$

is that there exists a generalised Bianchi tensor $\check{B}^a{}_{bcd}$ with respect to the symmetric connection Γ which satisfies, in addition to (2.10*a*) *and* (2.11),

$$h_{ai}\check{B}^{i}{}_{bcd}+h_{bi}\check{B}^{i}{}_{acd}=0 \tag{2.27}$$

when the matrix ${}^{s}\mathbf{A}_{\mathcal{B}}$ representing the mapping (2.18) *has maximal rank.*

The necessary condition is proved by constructing from the curvature tensor $R^a{}_{bcd}$ of the Weyl connection, the modified curvature tensor

$$\check{R}^a{}_{bcd}=R^a{}_{bcd}-\tfrac{1}{4}\delta^a_b R^i{}_{icd} \tag{2.28}$$

and noting that this tensor $\check{R}^a{}_{bcd}$ is a generalised Bianchi tensor with respect to the Weyl connection, satisfying (2.10a) and (2.11); it also—by virtue of (2.2)—satisfies the algebraic condition (2.27).

The curvature tensor $R^a{}_{bcd}$ of a symmetric connection also satisfies the sufficient as well as the necessary conditions and so we get the following corollary to theorem 1:

Corollary 2. *At any point in a co-ordinate domain U of* **M**, *a necessary and sufficient condition that the symmetric connection Γ is a Weyl connection, i.e., that Γ satisfies (for some metric h_{ab} and some vector λ_a)*

$$\nabla_c h_{ab} = h_{ab}\lambda_c \,, \tag{2.29}$$

is that the modified curvature tensor $\check{R}^a{}_{bcd}$ *of the symmetric connection* Γ, *defined by* (2.28), *satisfies*

$$h_{ai}\check{R}^i{}_{bcd} + h_{bi}\check{R}^i{}_{acd} = 0 \tag{2.30}$$

when the matrix ${}^s\mathsf{A}_{\check{R}}$ *representing the mapping*

$$\tilde{Q}_{cab} \to \check{R}^i{}_{b[cd}\tilde{Q}_{e]ai} + \check{R}^i{}_{a[cd}\tilde{Q}_{e]bi} \tag{2.31}$$

has maximal rank.

3. Conditions for semi-metric connections

To generalise the results in the previous section we shall assume that **M** is a smooth four-dimensional manifold on which there is an asymmetric connection Γ, associated covariant derivative ∇, and curvature tensor components $R^a{}_{bcd}$ defined in the usual way. The asymmetric connection can be split into its symmetric part with components $\Gamma^a_{(bc)}$ and its antisymmetric (torsion) part with components $T_{bc}{}^a (=2\Gamma^a_{[cb]})$, so that [#3]

$$\Gamma^a_{bc} = \Gamma^a_{(bc)} + \tfrac{1}{2}T_{cb}{}^a \,. \tag{3.1}$$

In addition on **M** there is defined a non-degenerate symmetric Lorentz metric with components h_{ab} which is not related *a priori* to the asymmetric connection Γ.

A semi-metric connection [4] is one in which the vector λ_a links the asymmetric connection Γ and metric h_{ab} according to

$$\nabla_c h_{ab} = h_{ab}\lambda_c \,. \tag{3.2}$$

The first integrability condition of (3.2) is easily seen to be

$$h_{ai}R^i{}_{bcd} + h_{bi}R^i{}_{acd} = 2h_{ab}(\nabla_{[d}\lambda_{c]} + \tfrac{1}{2}T_{cd}{}^i\lambda_i) \,. \tag{3.3}$$

We now wish to prove that, usually, this first set of integrability conditions (3.3) is *by itself* sufficient to imply (3.2). More generally, by analogy with the last section, we will first find sufficient conditions—in terms of the existence of a Bianchi tensor—for an asymmetric connection to be semi-metric, and then specialise the result to the curvature tensor $R^a{}_{bcd}$ of the asymmetric connection.

So we assume that there exists on **M** a "Bianchi tensor $B^a{}_{bcd}$ with respect to the

[#3] We define the curvature tensor of an asymmetric connection by $R^a{}_{bcd}V^b = (\nabla_c\nabla_d - \nabla_d\nabla_c - T_{cd}{}^b\nabla_b)V^a$. It should be noted that in refs. [4] and [8] the corresponding definition gives the indices of the curvature tensor as $R_{cdb}{}^a$. Our definition of torsion $T_{bc}{}^a$ given above agrees with the definition in ref. [4].

asymmetric connection Γ". This is a four-tensor $B^a{}_{bcd}$ which satisfies the same algebraic identities as the curvature tensor of an asymmetric connection [8],

$$B^a{}_{bcd}+B^a{}_{bdc}=0 , \tag{3.4a}$$

$$B^a{}_{[bcd]}+\nabla_{[b}T_{cd]}{}^a+T_{[bc}{}^iT_{d]i}{}^a=0 , \tag{3.4b}$$

and the same Bianchi-type equation as the curvature tensor of an asymmetric connection,

$$\nabla_{[e}B^a{}_{|b|cd]}+T_{[cd}{}^iB^a{}_{|b|e]i}=0 . \tag{3.5}$$

We further assume, in this case, that this Bianchi tensor $B^a{}_{bcd}$ satisfies an additional algebraic relation of the form

$$h_{ai}B^i{}_{bcd}+h_{bi}B^i{}_{acd}=h_{ab}b_{cd} , \tag{3.6}$$

where $b_{cd}(=-b_{dc})$ is a bivector defined on $\mathbf{M}$, which is easily seen to be given by

$$B^a{}_{acd}=2b_{cd} . \tag{3.7}$$

Making use of the Bianchi equation (3.5) we get

$$b_{cd}=\nabla_{[d}b_{c]}+\tfrac{1}{2}T_{cd}{}^ib_i . \tag{3.8}$$

b_a is of course defined up to an additive gradient.

If we now make the substitution

$$B^a{}_{bcd}=\check{B}^a{}_{bcd}+\tfrac{1}{4}\delta^a_bB^i{}_{icd} \tag{3.9}$$

into (3.6) and use (3.7), we obtain

$$h_{ai}\check{B}^i{}_{bcd}+h_{bi}\check{B}^i{}_{acd}=0 . \tag{3.10}$$

We note that $\check{B}^a{}_{bcd}$ shares the symmetry (3.4a) with $B^a{}_{bcd}$ satisfying

$$\check{B}^a{}_{bcd}+\check{B}^a{}_{bdc}=0 , \tag{3.11a}$$

and a more complicated version of (3.4b),

$$\check{B}^a{}_{[bcd]}=-\tfrac{1}{2}\delta^a_{[b}(\nabla_db_{c]}+\tfrac{1}{2}T_{cd]}{}^ib_i)-\nabla_{[b}T_{cd]}{}^a-T_{[bc}{}^iT_{d]i}{}^a . \tag{3.11b}$$

Also $\check{B}^a{}_{bcd}$ satisfies the Bianchi-type equation

$$\nabla_{[e}\check{B}^a{}_{|b|cd]}+T_{[cd}{}^i\check{B}^a{}_{|b|e]i}=0 \tag{3.12}$$

in the asymmetric connection Γ.

Let us now define a "generalised Bianchi tensor with respect to the asymmetric connection Γ" to be a four-tensor $B^a{}_{bcd}$ which satisfies the algebraic symmetry (3.4a) and also satisfies the Bianchi-type eq. (3.5) in the asymmetric connection Γ. The tensor $\check{B}^a{}_{bcd}$ defined by (3.9) is therefore—by virtue of (3.11a) and (3.12)—a generalised Bianchi tensor with respect to the asymmetric connection Γ.

We now follow the same procedure as in refs. [1,2] and the last section. Differentiating (3.10) and using the Bianchi equations (3.12) we find

$$\check{B}^i{}_{b[cd}\hat{Q}_{e]ai}+\check{B}^i{}_{a[cd}\hat{Q}_{e]bi}=0\,, \tag{3.13}$$

where

$$\hat{Q}_{cab}=-\nabla_c h_{ab}\,. \tag{3.14}$$

Again we will use the metric tensor h_{ab} to lower (and use h^{ab} to raise) indices—remembering that h_{ab} is not constant with respect to ∇.

The system (3.13) reduces—in four dimensions—to a square homogeneous system of thirty-six equations

$$\check{B}^i{}_{b[cd}\tilde{Q}_{e]ai}+\check{B}^i{}_{a[cd}\tilde{Q}_{e]bi}=0 \tag{3.15}$$

in the thirty-six unknowns

$$\tilde{Q}_{cab}=\hat{Q}_{cab}-\tfrac{1}{4}h_{ab}\hat{Q}_{ci}{}^{i}\,. \tag{3.16}$$

This means that the mapping

$$\tilde{Q}_{cab}\to\check{B}^i{}_{b[cd}\tilde{Q}_{e]ai}+\check{B}^i{}_{a[cd}\tilde{Q}_{e]bi} \tag{3.17}$$

can be represented by a 36×36 matrix ${}^{\mathrm{a}}A_{\mathrm{B}}$ consisting of components of the generalised Bianchi tensor $\check{B}^a{}_{bcd}$. If this matrix has maximal rank then the only solution to (3.17) is the trivial one,

$$\tilde{Q}_{cab}=0\,, \tag{3.18}$$

which is equivalent to

$$\nabla_c h_{ab}=-\tfrac{1}{4}h_{ab}\hat{Q}_{ci}{}^{i}\,. \tag{3.19}$$

So we arrive at the following result:

Theorem 3. *At any point in a co-ordinate domain U of* **M**, *a necessary and sufficient condition that an asymmetric connection Γ be a semi-metric connection, i.e., that Γ satisfies (for some metric h_{ab} and some vector λ_a)*

$$\nabla_c h_{ab}=h_{ab}\lambda_c\,, \tag{3.20}$$

is that there exists a generalised Bianchi tensor $\check{B}^a{}_{bcd}$ with respect to the asymmetric connection Γ which satisfies—in addition to (3.11a) and (3.12)—the algebraic constraint

$$h_{ai}\check{B}^i{}_{bcd}+h_{bi}\check{B}^i{}_{acd}=0 \tag{3.21}$$

when the matrix ${}^{\mathrm{a}}A_{\mathrm{B}}$ representing the mapping (3.17) has maximal rank.

The necessary condition is proved by constructing from the curvature tensor

$R^a{}_{bcd}$ of the semi-metric connection, the modified curvature tensor

$$\check{R}^a{}_{bcd} = R^a{}_{bcd} - \tfrac{1}{4}\delta^a_b R^i{}_{icd} \tag{3.22}$$

and noting that this tensor $\check{R}^a{}_{bcd}$ is a generalised Bianchi tensor with respect to the semi-metric connection, satisfying (3.11a) and (3.12). It also satisfies—by virtue of (3.3)—the algebraic condition (3.21).

We can easily deduce the special case of this theorem where the Bianchi tensor is specialised to the curvature tensor:

Corollary 4. *At any point in a co-ordinate domain U of* **M**, *a necessary and sufficient condition that the asymmetric connection Γ is a semi-metric connection, i.e., that Γ satisfies (for some metric h_{ab} and some vector λ_a)*

$$\nabla_c h_{ab} = h_{ab}\lambda_c\,, \tag{3.23}$$

is that the modified curvature tensor $\check{R}^a{}_{bcd}$ of the asymmetric connection Γ defined by (3.22) *satisfies*

$$h_{ai}\check{R}^i{}_{bcd} + h_{bi}\check{R}^i{}_{acd} = 0 \tag{3.24}$$

when the matrix ${}^{a}A_{\beta}$ representing the mapping

$$\tilde{Q}_{cab} \to \check{R}^a{}_{bcd}\tilde{Q}_{e]ai} + \check{R}^a{}_{bcd}\tilde{Q}_{e]bi} \tag{3.25}$$

has maximal rank.

4. Conditions for Riemann–Cartan connections

Riemann–Cartan connections [7] are generalisations of the metric connections discussed in refs. [1,2], and specialisations of the semi-metric connections of the last section; they are connections Γ which have an antisymmetric (torsion) part and which are also metric compatible, i.e.,

$$\nabla_c g_{ab} = 0\,. \tag{4.1}$$

From (3.3) it follows that the curvature tensor of such connections satisfies

$$g_{ai}R^i{}_{bcd} + g_{bi}R^i{}_{acd} = 0 \tag{4.2}$$

and hence the volume-preserving condition

$$R^a{}_{acd} = 0\,. \tag{4.3}$$

So we follow through the same argument as in the previous section, except that we begin from the assumption that our Bianchi tensor satisfies [in addition to (3.4a), (3.4b), (3.5)]

$$h_{ai}B^i{}_{bcd} + h_{bi}B^i{}_{acd} = 0 \tag{4.4}$$

and also that the connection is volume preserving, i.e., its curvature tensor satisfies (4.3).

For such connections our investigation does not have to stop at (3.19),

$$\nabla_c h_{ab} = -\tfrac{1}{4} h_{ab} \hat{Q}_{ci}{}^{i} , \tag{4.5}$$

as was the case for the more general semi-metric connections in the last section. The integrability condition of (4.5) gives

$$\tfrac{1}{2} h_{ab} (\nabla_{[c} \hat{Q}_{d]i}{}^{i} + \tfrac{1}{2} T_{dc}{}^{j} \hat{Q}_{ji}{}^{i}) = h_{ai} R^{i}{}_{bcd} + h_{bi} R^{i}{}_{acd} , \tag{4.6}$$

which implies—from (4.3)—that

$$\nabla_{[c} \hat{Q}_{d]i}{}^{i} + \tfrac{1}{2} T_{dc}{}^{j} \hat{Q}_{ji}{}^{i} = 0 , \tag{4.7}$$

and so in this case—as in refs. [1] and [2]—it follows that $\hat{Q}_{ai}{}^{i}$ is a gradient vector, so that

$$\nabla_c h_{ab} = h_{ab} \nabla_c \sigma . \tag{4.8}$$

Therefore there exists a metric g_{ab} which is annihilated by ∇,

$$g_{ab} = \mathrm{e}^{-\sigma} h_{ab} . \tag{4.9}$$

This establishes the following results:

Theorem 5. *At any point in a co-ordinate domain U of* **M**, *a necessary and sufficient condition that an asymmetric connection Γ which is volume preserving ($R^{a}{}_{acd}(\Gamma) = 0$) be a Riemann–Cartan connection is that there exists a non-degenerate metric h_{ab}* and a Bianchi tensor $B^{a}{}_{bcd}$ *with respect to the asymmetric connection Γ which satisfies—in addition to* (3.4*a*, *b*) *and* (3.5)*—the algebraic condition*

$$h_{ai} B^{i}{}_{bcd} + h_{bi} B^{i}{}_{acd} = 0 \tag{4.10}$$

when the matrix ${}^{a}A_{B}$ representing the mapping

$$Q_{cab} \to B^{i}{}_{b[cd} \tilde{Q}_{e]ai} + B^{i}{}_{a[cd} \tilde{Q}_{e]bi} \tag{4.11}$$

has maximal rank. Further, one possible metric g_{ab} for the connection Γ^{a}_{bc} is conformally related to the metric h_{ab} by

$$g_{ab} = \mathrm{e}^{-\lambda} h_{ab} , \tag{4.12}$$

where

$$\nabla_c \lambda = \tfrac{1}{4} h^{ab} \nabla_c h_{ab} . \tag{4.13}$$

As in the previous sections the necessary condition is trivially established by referring to the curvature tensor $R^{a}{}_{bcd}$.

For the special case of the curvature tensor the theorem gives

Corollary 6. *At any point in a co-ordinate domain U of* **M**, *a necessary and sufficient condition that an asymmetric connection Γ be a Riemann–Cartan connection is that there exists a non-degenerate metric h_{ab} so that the curvature tensor $R^a{}_{bcd}$ of the asymmetric connection Γ satisfies*

$$h_{ai}R^i{}_{bcd}+h_{bi}R^i{}_{acd}=0 \tag{4.14}$$

when the matrix ${}^aA_R$ representing the mapping

$$\tilde{Q}_{cab}\rightarrow R^i{}_{b[cd}\tilde{Q}_{e]ai}+R^i{}_{a[cd}\tilde{Q}_{e]bi} \tag{4.15}$$

has maximal rank. Further, one possible metric g_{ab} for the connection Γ^a_{bc} is conformally related to the metric h_{ab} by

$$g_{ab}=e^{-\lambda}h_{ab}\,, \tag{4.16}$$

where

$$\nabla_c\lambda=\tfrac{1}{4}h^{ab}\nabla_c h_{ab}\,. \tag{4.17}$$

Remark. By requiring, in the above theorem and corollary, that the connection be symmetric the original results for metric connections, in refs. [2] and [1], respectively, follow immediately.

5. The rank condition

As we pointed out in refs. [1] and [2] such results as those in the previous three sections are only meaningful if the rank condition on the respective matrices considered there is satisfied for significant classes of tensors. Although we have concentrated on four dimensions it is easily seen that the arguments in the previous sections apply equally in higher dimensions; the only difference being that the matrices are square only in four dimensions. However, it is only in four dimensional space–times that we have existing classification schemes which enable us to examine the rank condition in detail, and judge to what extent it is generally satisfied.

We showed in ref. [1]—for four-dimensional space–times—that *almost all* the curvature tensors of symmetric connections under discussion there satisfied the maximal rank condition. Because those curvature tensors had all the algebraic symmetries of a Riemann tensor we were able to use the standard decomposition and classification schemes associated with Riemann tensors [8], and by decomposing those curvature tensors into their Weyl and Ricci parts, we were able to see explicitly those situations where the rank condition failed. In particular we used the Petrov classification [8] of the Weyl curvature tensor to show that it is only when there are certain algebraic relations linking the (four real) Petrov scalars, and/or certain algebraic relations linking the Petrov scalars with the Ricci tensor components that the rank condition on the curvature tensor fails to be satisfied. Since the Bianchi tensor in ref. [2] had exactly the same algebraic structure as the curvature tensor in ref. [1] we were able to conclude, in that case also, that the rank condition only fails in the exactly analogous special circumstances.

It is important to note in ref. [2] that, although in the statement of the theorem we imposed a second index symmetry condition [of the type of (2.3b)] on the Bianchi tensor—and of course the curvature tensors also have this type of symmetry—this condition is not used in the actual proof of the theorem. So strictly the theorem in ref. [2] is still valid even if this condition is omitted. However, this symmetry is used in the discussion on the generality of the rank condition as outlined above. It is because of this additional symmetry that the Bianchi tensor and curvature tensor in refs. [1] and [2] can be decomposed (in the same way as a Riemann tensor) into only *two* parts—the Weyl and Ricci parts—which permits the Petrov classification to be used and enables us to draw the strong conclusions in refs. [1] and [2] on the very general nature of the rank condition, i.e., that it is satisfied for *almost all* of the Bianchi and curvature tensors, respectively.

Turning to theorem 1 in section 2 of this paper, the mapping (2.18) involves the *generalised* Bianchi tensor $\check{B}^a{}_{bcd}$. This tensor satisfies the usual conditions (2.9) and (2.10a) but does not satisfy the other condition, (2.3b). So—in a four-dimensional space–time—the generalised Bianchi tensor $\check{B}^a{}_{bcd}$ has more non-zero components than the Bianchi tensor $B^a{}_{bcd}$ in ref. [2], 36 compared to 20. This means that we cannot immediately carry through the same standard decomposition of $\check{B}^a{}_{bcd}$—into its Weyl and Ricci parts—as we were able to for $R^a{}_{bcd}$ and $B^a{}_{bcd}$ in the previous papers [1] and [2].

But there is an analogous decomposition—in a four-dimensional space–time, with respect to the metric h_{ab}—for any tensors like $\check{B}^a{}_{bcd}$ which satisfy the index symmetry (2.10a) and the condition (2.9) with respect to a metric h_{ab}. This is easiest described in spinor language [8,9]. The spinor counterpart of $\check{B}_{abcd}$ (where we use the metric tensor h_{ab} to lower the first index) can be decomposed into two spinors, X_{ABCD} and $\Phi_{ABA'B'}$, which because of (2.9) and (2.10a) have the properties

$$X_{ABCD}=X_{(AB)(CD)}\,, \quad \Phi_{ABC'D'}=\Phi_{(AB)(C'D')}\,. \tag{5.1}$$

The first of these complex spinors can be further decomposed into two symmetric spinors, Ψ_{ABCD} and Σ_{AB}, and a complex scalar Λ. (In the usual decomposition for the Riemann tensor, Σ_{AB} is identically zero, Λ is real and $\Phi_{ABA'B'}$ is hermitian. So for the generalised Bianchi tensor, the 3 complex components of Σ_{AB}, the imaginary component of Λ and the 9 imaginary components of $\Phi_{ABA'B'}$, carry the additional information of the extra 16 real components of $\check{B}_{abcd}$.) The completely symmetric spinor Ψ_{ABCD} has all the algebraic properties of the usual Weyl spinor with 5 complex components, and we shall refer to it again as the Weyl part. The other three spinors $\Phi_{ABA'B'}$, Σ_{AB} and Λ, are together known as the Ricci part of the generalised Bianchi tensor.

We can now use this decomposition to determine explicitly where the rank condition fails. First we consider the case when only the Weyl part Ψ_{ABCD} of $\check{B}_{abcd}$ is non-zero and we can perform a Petrov classification on this tensor in the usual way (making use of the metric h_{ab}), to conclude that it is only when there are certain very exceptional algebraic relations linking the (four real) Petrov scalars that the rank condition on the generalised Bianchi tensor fails to be satisfied. Next when we consider the remaining part (the Ricci part) of $\check{B}_{abcd}$ also to be non-zero, we can conclude that it is only when there exist certain very special algebraic relations linking together the Petrov scalars with components from the Ricci part, that the rank condition on the generalised Bianchi tensor fails to be satisfied. So, this time again as in ref. [2], *almost all* generalised Bianchi tensors satisfying (2.27) satisfy the rank condition.

For the case of the corollary in section 2, the modified curvature tensor does satisfy a second algebraic relation—of the type (2.10b); although it does not equate the 16 additional components to zero, the condition does exert considerable constraints in them. So when we follow through the same argument as in the last paragraph we have to take into account that some of the components of the Ricci part—those additional 16 real components—are already subject to some constraints; this reduces even further the number of modified curvature tensors satisfying the conditions of the theorem for which there is the possibility that the rank condition might fail.

(We could of course have required in the statement of theorem 1 the additional symmetry condition (2.10b) on the generalised Bianchi tensor $\check{B}^a{}_{bcd}$ and this would have made no difference to the proof of the sufficient part of the theorem; it would have—as in the curvature tensor case—slightly reduced the number of generalised Bianchi tensors satisfying the conditions of the theorem for which the rank condition might fail. However, on the other hand it would have put awkward restrictions on the choice of $\check{B}^a{}_{bcd}$, in any practical attempt to construct examples of the tensors $\check{B}^a{}_{bcd}$.)

A similar justification to that just given can be applied to the theorems and corollaries in sections 3 and 4 to show that the rank conditions there are also *almost always* satisfied.

6. Summary and discussion

We have found—in a four-dimensional space–time—that the class of asymmetric connections whose (modified) curvature tensors necessarily satisfy an algebraic relation of the type (1.2) also, generally, have such an algebraic relation as a sufficient condition. This is the class whose connections are known as semi-metric; this class includes metric, Weyl and Riemann–Cartan connections. We have also generalised these results to show how such connections depend on the existence of a Bianchi tensor—a tensor of a more general nature than a curvature tensor.

The algebraic condition (1.2) has been thoroughly investigated in ref. [10] for the case of a Riemann tensor (or, more generally, for any four-tensor which has all the index symmetries of a Riemann tensor). In ref. [2] we were able to make use of these existing results to obtain uniqueness results on the metric tensor in (1.2). However, in this paper the generalised Bianchi tensors and modified curvature tensors satisfying the algebraic condition (1.2) lack one of the symmetry properties, the symmetry in (1.5), used in obtaining the results in ref. [10], and so these results cannot be applied directly to the present investigations. An analysis of this more complicated algebraic condition—condition (1.2), excluding the property (1.5)—will be presented elsewhere, together with its application to the uniqueness of the metric of the metric-compatible connections considered in this paper.

Also in ref. [2] we examined the second set of integrability conditions alongside the first, and discovered that together they presented sufficient conditions for metric connections *for an even larger class of curvature tensors*—specifically for the class of curvature tensors which determine the metric in (1.2) uniquely (up to a conformal factor). A similar result is expected for the more general spaces considered in this paper.

Recently Hall [11] has studied, via holonomy groups, certain uniqueness problems associated with Weyl connections, and although the present form of these results are not immediately applicable to the results of this paper, clearly unifying the two approaches will bring a fuller understanding of this subject.

Finally we note that considerable work has been done, and continues to be done—from a variety of points of view—on the question of local existence of different types of connections [12–18]. However, in this paper we are concerned with only a specific and limited aspect of this very general and fundamental mathematical problem. We are not searching for a complete set of existence conditions which is valid for *all* possible spaces; even with restrictions on dimension and signature, such conditions will be complicated. Rather we are concerned with identifying a class of connections for which a simple algebraic condition on the curvature tensor of the form of (1.2) is a sufficient condition for their local existence. By restricting ourselves to four-dimensional space–times, we are able to answer this question in some detail. It is only in four dimensions that the crucial equation in our argument (2.16) gives a square system of equations and such a simple rank condition, and it is only in four-dimensional space–times that we are able to exploit the Petrov classification scheme to examine explicitly the rank condition and confirm that it is *almost always* satisfied. The motivation for requiring sufficient conditions in a form like (1.2) is that the analogous equation for symmetric metric connections has been studied extensively in refs. [1,2,10,15] and has enabled a lot of information about the relationships between curvature tensors, connections and metrics to be obtained, without having to consider explicitly the complicated differential equations which relate the curvature tensor with metric or connection. In addition Ihrig [15] has outlined a procedure, using eq. (1.2), which enables metrics to be calculated from their Riemann tensors *by purely algebraic means*; and McIntosh and Halford [19] have shown how this procedure can work in practice. We intend to generalise this procedure to the more general classes of spaces discussed here.

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